Brunnermeier&Sannikov(2014) model derivation

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Note 1. *There are omitted predetermined setting and adding "*" means as the same in Brunnermeier and Sannikov(2014)*

Suppose we have known the state variables, q_t and K_t

$$\mathrm{d}k_t = (\Phi(\iota_t) - \delta)k_t\mathrm{d}t + \sigma k_t \mathrm{d}Z_t \tag{1}$$

$$\mathrm{d}q_t = \mu_t^q q_t\mathrm{d}t + \sigma_t^q q_t \mathrm{d}Z_t. \tag{2}$$

Based on the Ito's Product rule of geometry Brownian Motion:

$$\frac{\mathrm{d}(k_t q_t)}{k_t q_t} = (\Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q)dt + (\sigma + \sigma_t^q)dZ_t \tag{3}$$

$$\frac{\mathrm{d}n_t}{n_t} = x_t\mathrm{d}r_t^k + (1-x_t)r\mathrm{d}t - \frac{\mathrm{d}c_t}{n_t} \tag{4}$$

N_t denotes the sum of all experts, and

$$\frac{\mathrm{d}n_t}{n_t} = x_t\mathrm{d}r_t^k + (1-x_t)r\mathrm{d}t - \frac{\mathrm{d}c_t}{n_t}$$

and $X_t = x_t = \psi(q_t K_t/N_t) = \frac{\psi}{\eta}$ postulate the state equation of N_t. Let $N_t = m \cdot n_t$

$$\begin{aligned}\mathrm{d}N_t = d(m\cdot n_t) &= mn_t x_t\mathrm{d}r_t^k + mn_t(1-x_t)r\mathrm{d}t - m\mathrm{d}c_t \\ &= N_t x_t \mathrm{d}r_t^k + N_t(1-x_t)r\mathrm{d}t - \mathrm{d}C_t \\ &= N_t\psi q_t K_t(\mathrm{d}r_t^k - r\mathrm{d}t) + N_t r\mathrm{d}t - \mathrm{d}C_t\end{aligned} \tag{5}$$

where $\frac{d(k_t q_t)}{k_t q_t}$ is the expert return on capital at time t. And $\frac{d(q_t K_t)}{q_t K_t}$ represents the income level of all asset holders at time t, where Expert and Household account for ψ,$1-\psi$ respectively. We notice that the total repay including the part of dividend. Expert's total return dr_t^k is

$$\begin{aligned} \mathrm{d}r_t^k &= \frac{a-\iota}{q_t}dt + \frac{d(k_t q_t)}{k_t q_t} \\ &= \frac{a-\iota}{q_t}\mathrm{d}t + (\Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q)\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t \end{aligned} \tag{6}$$

Similarly, we can get the total return rate $d\underline{r}_t^k$ belonging to household

$$\begin{aligned} \mathrm{d}\underline{r}_t^k &= \frac{\underline{a}-\underline{\iota}}{q_t}dt + \frac{\mathrm{d}(k_t q_t)}{k_t q_t} \\ &= \frac{\underline{a}-\underline{\iota}}{q_t}dt + (\Phi(\underline{\iota}_t) - \underline{\delta} + \mu^q + \sigma\sigma_t^q)\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t \end{aligned} \tag{7}$$

do the partial derivative of expert's and household's return ((6) and (7)) relative to ι such that we can get the optimal condition as follows.

$$\begin{aligned} \max_{\iota} \Phi(\iota) - \frac{\iota}{q_t} \\ F.O.C. \Rightarrow \Phi'(\iota) = \frac{1}{q_t} \\ \iota_t = \underline{\iota_t} = \iota(q_t) \end{aligned}$$

Therefore, we can combine the optimal investment rate ι derived from (6) and (7). But the original parameters a, δ stay the same. The physical assets held by household are

$$1 - \psi_t = \frac{1}{K_t}\int_{\mathbb{J}} \underline{k_t^j}\mathrm{d}j$$

Optimal Asset Selection

Under the equilibrium condition, the expectation return of household will no more than free-risk rate. i.e.

$$E_t[\mathrm{d}\underline{r}_t^k]/dt = \frac{\underline{a}-\iota}{q_t} + \Phi(\iota_t) - \underline{\delta} + \mu^q + \sigma\sigma_t^q \le r, \quad 1 - \psi_t > 0 \tag{H}$$

If the households' return over r i.e risk-free rate then household will not lend to experts rather than increasing the investment rate which cutting off the return in the meanwhile.

We need to consider the sum of infinite expected utility of expert to infer the optimal decision function.

$$\theta_t n_t \equiv E_t\left[\int_t^{\infty} e^{-\rho(s-t)} dc_s\right] \tag{10*}$$

where θ_t is the stochastic opportunities set. Furthermore, we consider such a process,

$$\Theta_t = \int_0^t e^{-\rho s} n_s d\zeta_s + e^{-\rho t}\theta_t n_t$$

employing the integration by parts formula

$$\mathrm{d}\Theta_t = e^{-\rho t}(n_t \mathrm{d}\zeta_t - \rho\theta_t n_t \mathrm{d}t + \mathrm{d}(\theta_t n_t)) \tag{8}$$

from $E(\mathrm{d}\Theta_t) = 0$ we could know that Θ_t is a martingale. And the drift term of $d\Theta_t$ equals to zero based on the property of martingale. plug into (10*)

$$\rho\theta_t n_t = n_t d\zeta_t + E[d(\theta_t n_t)]$$

when n_t follows previous equation, also this economic system satisfies the transversality condition $E(e^{-\rho t}\theta_t n_t) \to 0$. Further, it can be proved that the Bellman equation is established by inverse proof, and the optimal strategy under constraints can be obtained:

$$\begin{aligned} \rho\theta_t n_t dt &= \max_{\hat{x}_t \geq 0, d\hat{\zeta} \geq 0} n_t d\hat{\zeta}_t + E[d(\theta_t n_t)] \\ s.t. \quad \frac{dn_t}{n_t} &= \hat{x}_t dr_t^k + (1-\hat{x}_t) r dt - d\hat{\zeta}_t \end{aligned} \tag{9}$$

Proposition 1. *Separate the above state variable processes and consider the stochastic process of θ.*

$$\frac{d\theta}{\theta} = \mu_t^\theta dt + \sigma_t^\theta dZ_t.$$

(i) $\theta \geq 1$, and only consume when $\theta = 1$
(ii) μ_t^θ is the drift term of θ at time t.

$$\mu_t^\theta = \rho - r \tag{E}$$

(iii) any $x_t > 0$

$$\frac{a - \iota(q_t)}{q_t} + \Phi(\iota(q_t)) - \delta + \mu^q + \sigma\sigma_t^q - r = -\sigma_t^\theta(\sigma + \sigma_t^q) \tag{EK}$$

when $x_t = 0$, $E[dr_t^k]/dt - r \leq -\sigma_t^\theta(\sigma + \sigma_t^q)$
(iv) The transversality $E[e^{-\rho t}\theta_t n_t] \to 0$

Proof. According to the hypothesis, Hamilton equation is constructed:

$$\begin{aligned} H &= n_t \mathrm{d}\hat{\zeta}_t + E[d(\theta_t n_t)] + \theta_t(n_t \mu_t^n) + \sigma_t^\theta \theta_t (n_t \sigma_t^n) \\ &= n_t \mathrm{d}\hat{\zeta}_t + e^{-\rho t}(n_t \mathrm{d}\hat{\zeta}_t - \rho\theta n_t) + \theta_t\left(-n_t \mathrm{d}\hat{\zeta}_t + n_t \hat{x}_t(\frac{a-\iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q) + n_t(1-\hat{x}_t) r\right) \\ &+ \sigma_t^\theta \theta_t (n_t \hat{x}_t (\sigma + \sigma_t^q)) \end{aligned}$$

where θ and σ^θ is the co-states and

$$E[\mathrm{d}(\theta_t n_t)] = e^{-\rho t}(n_t \mathrm{d}\hat{\zeta}_t - \rho\theta n_t)$$

$$dr_t^k = (\frac{a-\iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q)dt + (\sigma + \sigma_t^q)dZ_t \tag{10}$$

$$\frac{\mathrm{d}\theta_t}{\theta_t} = \mu_t^{\theta_t}\mathrm{d}t + \sigma^{\theta_t}\mathrm{d}Z_t = -r\mathrm{d}t - \sigma_t^\theta \mathrm{d}Z_t \tag{11}$$

where the first order conditions of $\iota_t, \mathrm{d}\hat{\zeta}_t, \hat{x}_t$ are

$$\Phi'(\iota_t) = \frac{1}{q_t} \tag{12}$$

$$e^{-\rho t} = \theta_t - 1 \tag{13}$$

$$\frac{a-\iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q - r = -\sigma_t^\theta(\sigma + \sigma_t^q). \tag{14}$$

And θ_t satisfies BSDE condition:

$$\mathrm{d}\theta_t = -\frac{\partial H}{\partial n_t}\mathrm{d}t - \sigma_t^\theta\theta_t\mathrm{d}Z_t \tag{15}$$

where drift term θ_t is

$$\begin{aligned}\mu_t^\theta\theta_t &= \frac{\partial H}{\partial n_t} \\ &= -\theta_t\left[\hat{x}_t(\frac{a-\iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q) + (1-\hat{x}_t)r + \hat{x}_t\sigma_t^\theta(\sigma+\sigma_t^q)\right] + \mathrm{d}\hat{\zeta}_t + e^{-\rho t}(\mathrm{d}\hat{\zeta}_t - \rho\theta) \\ &= -r\theta_t + \rho\theta_t\end{aligned}$$

Thus the law motion of θ_t is as follows

$$\frac{\mathrm{d}\theta_t}{\theta_t} = (\rho - r_t)\mathrm{d}t + \sigma^\theta\mathrm{d}Z_t$$

when $\hat{x}_t = 0$ the corresponding F.O.C. transfer to loose constraint

$$\frac{a-\iota}{q_t} + \Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q \leq -\sigma_t^\theta(\sigma + \sigma_t^q).$$

Using the first-order condition (F.O.C.) of $d\hat{\zeta}$, we can determine that θ_t is greater than or equal to 1. As θ_t approaches 1, the calculated value of $e^{-\rho t}$ approaches 0, indicating that the discounted value of future consumption in the current period tends towards 0. Therefore, the expert will opt for current consumption. □

Wealth Distribution

Combining equation (6) with equation (7) using Ito formula, we obtain the following formula:

$$\begin{aligned}\frac{\mathrm{d}(q_tK_t)}{q_tK_t} &= \mathrm{d}r_t^k - \frac{a-\iota(q_t)}{q_t}\mathrm{d}t - (1-\psi)(\underline{\delta}-\delta)\mathrm{d}t \\ &= \frac{\mathrm{d}(k_tq_t)}{k_tq_t} - (1-\psi)(\underline{\delta}-\delta)\mathrm{d}t \\ &= (\Phi(\iota_t) - \delta + \mu^q + \sigma\sigma_t^q - (1-\psi)(\underline{\delta}-\delta))dt + (\sigma+\sigma_t^q)\mathrm{d}Z_t\end{aligned} \tag{16}$$

In mathematical terms, η_t represents the proportion of an expert's wealth to the entire asset.

$$\eta_t \equiv \frac{N_t}{q_tK_t} \in [0,1]$$

Using the equation (16), we can determine the motion law for $d(\frac{1}{q_tK_t})$.

$$\frac{\mathrm{d}(\frac{1}{q_tK_t})}{\frac{1}{q_tK_t}} = -\mathrm{d}r_t^k + \frac{a-\iota(q_t)}{q_t}\mathrm{d}t + (1-\psi)(\underline{\delta}-\delta)\mathrm{d}t + (\sigma+\sigma_t^q)^2\mathrm{d}t \tag{17}$$

By combining equation (5) with (17), we can determine the state variable of η_t.

$$\begin{aligned}\mathrm{d}(\frac{1}{q_tK_t}) &= \frac{1}{q_tK_t}\left(-\mathrm{d}r_t^k + \frac{a-\iota(q_t)}{q_t}\mathrm{d}t + (1-\psi)(\underline{\delta}-\delta)\mathrm{d}t + (\sigma+\sigma_t^q)^2\mathrm{d}t\right) \\ \mathrm{d}N_t &= N_t\left(\psi q_tK_t(\mathrm{d}r_t^k - r\mathrm{d}t) + r\mathrm{d}t - \mathrm{d}\zeta_t\right) \\ \mathrm{d}\eta_t &= (\mathrm{d}N_t)\frac{1}{q_tK_t} + N_t\mathrm{d}(\frac{1}{q_tK_t}) + \psi q_tK_t(\sigma+\sigma_t^q)\frac{-1}{q_tK_t}(\sigma+\sigma_t^q)\mathrm{d}t \\ &= \eta_t\left(\psi(\frac{1}{\eta_t}-1)(\mathrm{d}r_t^k - r\mathrm{d}t) - \mathrm{d}\zeta_t + \frac{a-\iota(q_t)}{q_t}\mathrm{d}t + (1-\psi)(\underline{\delta}-\delta)\mathrm{d}t\right) + (\eta-\psi)(\sigma+\sigma_t^q)^2 \\ &= (\psi_t-\eta_t)(\mathrm{d}r_t^k - r\mathrm{d}t - (\sigma+\sigma_t^q)^2\mathrm{d}t) + \eta_t\left(\frac{a-\iota_(q_t)}{q_t} + (1-\psi_t)(\underline{\delta}-\delta)\right)\mathrm{d}t - \eta_t\mathrm{d}\zeta_t\end{aligned} \tag{14*}$$

Based on equation (EK) and the value of dr_t^k, we can deduce:

$$\mathrm{d}r_t^k - (\sigma+\sigma_t^q)\mathrm{d}Z_t = -\sigma_t^\theta(\sigma+\sigma_t^q)\mathrm{d}t + r\mathrm{d}t$$

and transfer the above formula to

$$\mathrm{d}\eta_t/\eta_t = \frac{\psi_t-\eta_t}{\eta_t}\left(-(\sigma+\sigma_t^q)(\sigma_t^\theta+\sigma+\sigma_t^q) + \frac{a-\iota_(q_t)}{q_t} + (1-\psi_t)(\underline{\delta}-\delta)\right)\mathrm{d}t - \mathrm{d}\zeta_t + \frac{\psi_t-\eta_t}{\eta_t}(\sigma+\sigma_t^q)\mathrm{d}Z_t \tag{18}$$

Thus, the drift of $\mathrm{d}\eta_t$

$$\mu_t^\eta = -\sigma_t^\eta(\sigma+\sigma_t^q+\sigma_t^\theta) + \frac{a-\iota(q_t)}{q_t} + (1-\psi_t)(\underline{\delta}-\delta) \tag{19}$$

The volatility term is

$$\sigma_t^\eta = \frac{\psi_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q) \tag{20}$$

Markov Equilibrium

Use Ito's Lemma to $q(\eta_t)$

$$\mathrm{d}q(\eta) = (q'(\eta)(u_t^\eta \eta) + \frac{1}{2}q''(\eta)(\sigma_t^\eta \eta)^2)\mathrm{d}t + \sigma_t^\eta q'(\eta)\eta dZ_t \tag{21}$$

corresponding term to the Brownian term of $\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t$

$$q(\eta)\sigma_t^q = \sigma_t^\eta q'(\eta)\eta \tag{22}$$

(20) plug in above equation we will get

$$\sigma_t^q q(\eta) = q'(\eta)(\psi - \eta)(\sigma + \sigma_t^q) \quad \Rightarrow \quad \sigma_t^q = \frac{q'(\eta)}{q(\eta)} \frac{(\psi - \eta)\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\psi - \eta)}. \tag{23}$$

In the meanwhile from (EK) and (H) we can infer (C2)

$$\frac{a - \iota(q_t)}{q_t} + \Phi(\iota(q_t)) - \delta + \mu^q + \sigma\sigma_t^q + \sigma_t^\theta(\sigma + \sigma_t^q) = r \tag{EK}$$

$$\frac{\underline{a} - \iota(q_t)}{q_t} + \Phi(\iota(q_t)) - \underline{\delta} + \mu^q + \sigma\sigma_t^q \le r \tag{H}$$

$$\frac{a - \underline{a}}{q(\eta)} + \underline{\delta} - \delta + (\sigma + \sigma_t^q)\sigma_t^\theta \ge 0 \tag{C2}$$

when $\psi < 1$ or $1 - \psi > 0$ i.e. household holds physical asset as well. When $q(\eta), q'(\eta), \theta(\eta) > 0$ and $\theta'(\eta) < 0$, we can infer that $\sigma_t^q, \sigma_t^\eta > 0$ will go up with ψ and $\sigma_t^\theta < 0$ goes down with ψ. $q(\eta)$ and ψ are unrelated. Only consider the part of $(\sigma + \sigma_t^q)\sigma_t^\theta$. Thus the inequality (C2)'s LHS will go down with ψ.

parameter solutions

from the previously synthetic conditions we can infer that:

$$\begin{aligned}
\mu_t^\eta &= -\sigma_t^\eta(\sigma + \sigma_t^q + \sigma_t^\theta) + \frac{a - \iota(q(\eta))}{q(\eta)} + (1 - \psi)(\underline{\delta} - \delta), \\
\mu_t^q &= r - \frac{a - \iota(q(\eta))}{q(\eta)} - \Phi(q(\eta)) + \delta - \sigma\sigma_t^q - \sigma_t^\theta(\sigma + \sigma_t^q), \\
\mu_t^\theta &= \rho - r
\end{aligned} \tag{19*}$$

Similar with (21) we can derivative $\theta(\eta)$ employing Ito Lemma:

$$\mathrm{d}\theta(\eta) = \left(\mu_t^\eta \theta'(\eta) + \frac{1}{2}(\eta\sigma_t^\eta)^2\theta''(\eta)\right)\mathrm{d}t + \theta'(\eta)\sigma_t^\eta \eta \mathrm{d}Z_t \tag{24}$$

use the corresponding term with the Brownian Term in $\frac{\mathrm{d}\theta}{\theta} = \mu_t^\theta \mathrm{d}t + \sigma_t^\theta \mathrm{d}Z_t$

$$\sigma_t^\theta = \frac{\theta'(\eta)}{\theta(\eta)}\sigma_t^\eta \eta \tag{25}$$

Take into account the corresponding volatility term.

$$\theta(\eta)\mu_t^\theta = (\eta\mu_t^\eta)\theta(\eta)' + \frac{1}{2}(\sigma_t^\eta\eta)^2\theta(\eta)'' \Rightarrow \theta(\eta_t)''(\eta) = 2(\theta(\eta)\mu_t^\theta - \theta(\eta)'\mu_t^\eta)/(\sigma_t^\eta\eta)^2 \tag{26}$$

$$q(\eta)\mu_t^q = (\eta\mu_t^\eta)q(\eta)' + \frac{1}{2}(\sigma_t^\eta\eta)^2 q(\eta)'' \Rightarrow q(\eta_t)''(\eta) = 2(q(\eta)\mu_t^q - q(\eta)'\mu_t^\eta)/(\sigma_t^\eta\eta)^2 \tag{27}$$

You can get it with a little transformation:

$$q''(\eta) = \frac{2[\mu_t^q q(\eta) - q'(\eta)\mu_t^\eta \eta]}{(\sigma_t^\eta)^2\eta^2} \quad \textit{and} \quad \theta''(\eta) = \frac{2[\mu_t^\theta \theta(\eta) - \theta'(\eta)\mu_t^\eta \eta]}{(\sigma_t^\eta)^2\eta^2} \tag{19*}$$

Boundary condition

1. under $\eta = 0$ i.e. household hold all asset, which $q(0) = \underline{q}$ guarantees
2. proposition1 when $\theta = 1$ expert choose consuming. we denote η^* as the consumption point, thus $\theta(\eta^*) = 1$
3. $q'(\eta^*) = 0, \theta'(\eta^*) = 0$ means the economy is stable. If $q'(\eta^*) < 0$ will make the experts holding assets suffer loss. If $\theta'(\eta^*) < 0$ would violate proposition1.
4. $\theta(0) = \infty$

By combining the above conditions, we can solve the unknown boundary PDE problem. The specific method is described in the article. The equilibrium domain is $[0, \eta^*]$, and the function boundary is as follows:

$$q(0) = \underline{q}, \quad \theta(\eta^*) = 1, \quad q'(\eta^*) = 0, \quad \theta'(\eta^*) = 0, \quad \textit{and} \quad \lim_{\eta \to 0} = \infty \tag{28}$$

make (C2) satisfied, i.e. the return rate of household in (H) equals to free-risk rate. expert also choose consuming.

$$\frac{a - \underline{a}}{q(\eta)} + \underline{\delta} - \delta + (\sigma + \sigma_t^q)\sigma_t^\theta = 0, \tag{C2*}$$

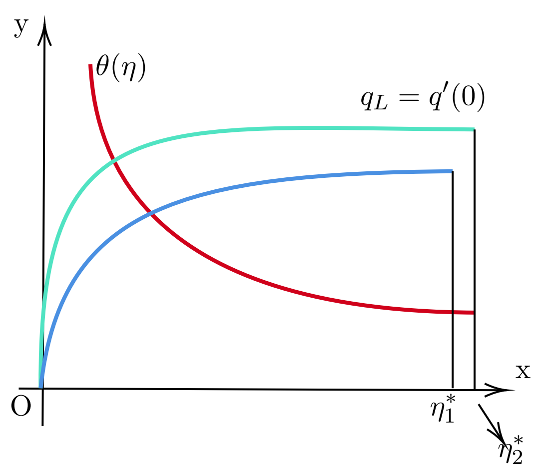
where

$$\sigma_t^\eta \eta = \frac{(\psi - \eta)\sigma}{1 - (\psi - \eta)q'(\eta)/q(\eta)}, \quad \sigma_t^q = \frac{q'(\eta)}{q(\eta)}\sigma_t^\eta \eta \quad \textit{and} \quad \sigma_t^\theta = \frac{\theta'(\eta)}{\theta(\eta)}\sigma_t^\eta \eta \tag{12-14}$$

in the domain $\psi \in (\eta, \eta + q(\eta)/q'(\eta)$ find η which satisfies (C2*) if the found point out of boundary i.e. $\psi > 1$ which means the $\eta^{*'}$ in $\psi'(\eta_t^{*'})$ less than η^* in $\theta(\eta^*)$ and $q(\eta^*)$. Let $\psi = 1$ and calculate (12-14) to get $\theta(\eta)$ and $q(\eta)$ again. At last, calculate eq (19*)

Detailed algorithm

- To begin with, since $\eta = 0$ is a singular point, we must determine the endogenous η to match the boundary condition of 0 and η^*.
- Next, we continue with (28) and set the initial condition as $q(0) = \underline{q}$, $\theta(0) = \theta(\eta^*) = 1$, and $\theta(0) = -10^{10}$.
- We assume that $q_L = 0$ and $q_H = 10^{15}$.
- We set $q'(0)$ to be $(q_L q_H)/2$ as a condition to enter the solution of the ordinary differential equation to obtain $q(\eta)$ and $\theta(\eta)$. Each calculation of the ordinary differential equation can be further calculated to get different η^*. We check if any one of them meets its boundary conditions. If $q'(\eta)$ reaches zero, it indicates that the set of initial conditions is falling too fast. We let $q_L = q'(0)$ if it is not the case that $q'(\eta)$ arrives first. Otherwise, we make $q_H = q'(0)$.
- After multiple iterations, convergence occurs when $q_L = q_H$.
- Finally, we need to make an adjustment of $\theta(\eta^*)$ such that $\theta(\eta^*) = 1$.



solve ψ

Enter the loop and put the updated initial value condition into the function for iterative solution. in fnct, we need to get the value of Φ and ι from investment fnct. Set the left- right- boundary of ψ

to η and $\min(1, \eta + q(\eta)/q'(\eta))$ respectively. Iterating in the iterator, also let $\psi = (\psi_L \psi_R)/2$, and then solve under the given function.

$$amp = 1 - q'(0)(\psi - \eta)/q(0) \tag{29}$$
$$\sigma^\eta \eta = \sigma(\psi - \eta)/amp \tag{30}$$
$$\sigma^\theta = \sigma^\eta \eta \theta'(0)/\theta \tag{31}$$
$$rp = -\sigma^\theta(\sigma + \sigma^q) \tag{32}$$
$$hp = (\underline{a} - a)/q + \delta - \underline{\delta} + rp \tag{33}$$

When the risk premium of a household is greater than 0, it means that the household will hold more assets. This will eventually reach equilibrium as ψ changes, resulting in a scalar ψ. Once solved, you can input the solved ψ into $\sigma^\eta \eta$ and $\sigma^\eta \eta$ into σ^θ, σ^q. Additionally, you can solve for expressions of μ^q and μ^η.

$$\mu^q = r - (a - \iota)/q(0) - \Phi + \delta - \sigma\sigma^q + rp \tag{34}$$
$$\mu^\eta \eta = -(\psi - \eta)(\sigma + \sigma^q(0))(\sigma + \sigma^q + \sigma^\theta) + \eta(a - \iota)/q(0) + \eta(1 - \psi)(\underline{\delta} - \delta) \tag{35}$$
$$q'' = 2((\mu^q q(0) - q'(0))\mu^\eta \eta)/\sigma^\eta \eta^2 \tag{36}$$
$$\theta'' = 2((\rho - r)\theta(0) - \theta'(0)) \tag{37}$$

First, solve the ordinary differential equation about $q(\eta)$ using $q''(\eta)$, and then insert the resulting $q(\eta)$ expression into θ'' to solve.